# An Introduction to Lie 3-Algebra

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# Nambu's Generalization of Canonical Formulation

- Rigid rotator as an example [Nambu 73]
- Hamilton-Jacobi eqs:

  $\{H, x\} = \mathrm{d}x/\mathrm{d}t$, $\{H, p\} = \mathrm{d}p/\mathrm{d}t$
- $\{H, F(y)\} = \mathrm{d}F(y)/\mathrm{d}t$, $y = (x, p)$
- Generalization:

  $\{H, G, F(z)\} = \mathrm{d}F(z)/\mathrm{d}t$, $z = (p, q, r)$

  for a 3 dim. phase space
- Conservation of phase space volume
- Invariant under canonical transfs. of $H$, $G$.

# Generalization of Noncommutative Geometry

- Noncommutative space:

  $[x, y] = i\theta$,  approx. by $\{x, y\} = \theta$

- Generalization (nonassociative space?):

  $\{x, y, z\} = \theta$  for a 3 dim. space

- Physical laws deformed
- Implications to Cosmology?
- String Theory / M Theory:

  *B* field on D-brane → noncommutativity

  *C* field on M5-brane → Nambu-Poisson bracket

# Lie (2-)algebra

- $H = \{ A = \Sigma_{a=1\sim n} A_a T^a \}$, $[A, B] \in H$
- $[A, B]$ = bi-linear, skew-symmetric
- $[A, [B, C]] = [[A, B], C] + [B, [A, C]]$ ← Covariance
- $[A, [B, C]] - [B, [A, C]] = [[A, B], C]$ ← Closure

  (The same Jacobi id. written differently.)
- Let $\delta_A B = [A, B]$

  $\delta_A [B, C] = [\delta_A B, C] + [B, \delta_A C]$

  $\delta_1 \delta_2 B - \delta_2 \delta_1 B = [\delta_1, \delta_2 ]B$

# Commutator vs. Lie bracket

- Commutator [A, B] = A.B - B.A
- An *associative* algebra (rules of multiplication that satisfy associativity) is assumed.
- Jacobi identity is a result of associativity: (A.B).C = A.(B.C)
- Commutator = Lie bracket for representations of Lie algebra using matrices.
- The algebra is called the "universal enveloping Lie algebra".

# Lie 3-algebra

- $H = \{A=\Sigma_{a=1\sim n} A_a T^a\}$, $[A, B, C] \in H$
- $[A, B, C]$ = tri-linear, skew-symmetric
- $[A, B, [C, D, E]] = [[A, B, C], D, E] +$

  $[C, [A, B, D], E] + [C, D, [A, B, E]]$

  fundamental id. (generalized Jacobi id.)
- Let $\delta_{AB} C = [A, B, C]$

  $\delta_{AB} [C, D, E] = [\delta_{AB} C, D, E] + [C, \delta_{AB} D, E]$

  $+ [C, D, \delta_{AB} E]$

# $\mathcal{A}_4$

- 4 generators $\{T_a\}$ (a = 1,2,3,4)
- $[T_a, T_b, T_c] = \varepsilon_{abcd}\, T_d$

  [Filippov 85: n-Lie algebras]

- generalization of *su(2)*
- Symmetry transformation is generated by two generators:

  $\delta_\Lambda A = \Lambda_{ab}\, [T_a, T_b, A]$

  = infinitesimal SO(4) rotation

# Poisson bracket

- Poisson brackets are infinite dim. Lie algs.
- $\{f(x), g(x)\} = P^{ab}(x)\, \partial_a f(x)\, \partial_b g(x)$

1. Skew-symmetry
2. Jacobi identity
3. Leibniz rule (new addition):

   $\{f, gh\} = \{f, g\}h + g\{f, h\}$

- Darboux theorem:

  Locally $P^{ab}$ is canonical for 2m of the $n$ coordinates

# Nambu-Poisson bracket

- Generalization of Poisson brackets

  [Nambu 73, Takhtajan 94]

- $\{f, g, h\} = P^{abc}\, \partial_a f\, \partial_b g\, \partial_c h$

1. Skew-symmetry
2. Fundamental identity
3. Leibniz rule:

   $\{f, g, h_1.h_2\} = \{f, g, h_1\}h_2 + \{f, g, h_2\}h_1$

- Decomposability theorem:

  Locally $P^{abc} = \varepsilon^{abc}$ for 3 of the $n$ coordinates

# Phenomenology of Nonassociative Geometry

- Nonassociative space:

  $\{x, y, z\} = \theta$ for our dim. space

- Rotation and translation symmetry preserved
- Gauge transformation laws deformed
- Implications to Cosmology?

  Modification of CMB spectrum?

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